

A new method for extracting poles from single-channel data based on Laurent expansion of T-matrices with Pietarinen power series representing the non-singular part

Alfred Švarc*

Rudjer Bošković Institute, Bijenička cesta 54, P.O. Box 180, 10002 Zagreb, Croatia

Mirza Hadžimehmedović, Hedim Osmanović, and Jugoslav Stahov

University of Tuzla, Faculty of Science, Univerzitetska 4, 35000 Tuzla, Bosnia and Herzegovina

(Dated: December 21, 2012)

We present a new approach to quantifying pole parameters of single-channel processes based on Laurent expansion of partial wave T-matrices. Instead of guessing the analytical form of non-singular part of Laurent expansion as it is usually done, we represent it by the convergent series of Pietarinen functions. As the analytic structure of non-singular term is usually very well known (physical cuts with branchpoints at inelastic thresholds, and unphysical cuts in the negative energy plane), we show that we need one Pietarinen series per cut, and the number of terms in each Pietarinen series is automatically determined by the quality of the fit. The method is tested on a toy model constructed from two known poles, various background terms, and two physical cuts, and shown to be robust and confident up to three Pietarinen series. We also apply this method to Zagreb CMB amplitudes for the N(1535) 1/2- resonance, and confirm the full success of the method on realistic data. This formalism can also be used for fitting experimental data, and the procedure is very similar as when Breit-Wigner functions are used, but with one modification: Laurent expansion with Pietarinen series is replacing the standard Breit-Wigner T-matrix form.

PACS numbers: 11.55.-m, 11.55.Fv, 14.20.Gk, 25.40.Ny.

The recent Camogli workshop [1] has finally inaugurated the fact that poles, and not Breit-Wigner parameters determine and quantify resonance properties, and that they should be used as a link between scattering theory and QCD. However, at the same time, the question of finding an adequate procedure to extract them from single-channel T-matrices has been opened. Experimentalists are quite familiar with fitting the data with Breit-Wigner functions (either with constant parameters and very general backgrounds, or with energy dependent mass and width), but are inexperienced when actual complex energy poles have to be used. A simple procedure for pole extraction is still missing. Up to now poles have usually been extracted from theoretical single or multi-channel models fitted to the data using either of standard pole extraction methods: analytic continuation of the model functions into the complex energy plane [2–6], speed plot [7], time delay [8], N/D method [9], regularization procedure [10], etc. However, this required solving standardly elaborated single/coupled-channel models and analyzing the obtained analytic solution which implicitly contained both parts: singular and background. Hence, the analytic form of the full solution varied from model to model; pole-background separation was not unique, and this introduced uncertainties in pole extraction procedures. The intention of this paper is to offer simple, robust and confident method how to obtain scattering matrix poles, but maximally avoid the need to

speculate about type and form of background terms. We base our analysis on Laurent expansion of partial wave T-matrices which uniquely separates singular from finite terms, and treat singular and finite terms separately. Our main assumption is that all scattering matrix poles are of the first order.

The starting point of our method is Laurent expansion for a function with one pole:

$$T(\omega) = \frac{a_{-1}}{\omega_0 - \omega} + \tilde{B}^L(\omega); \quad a_{-1}, \omega_0, \omega \in \mathbb{C}. \quad (1)$$

where a_{-1} and ω_0 are residuum and pole position respectively, and function $\tilde{B}^L(\omega) = \sum_{n=0}^{\infty} a_n(\omega_0 - \omega)^n$ is regular everywhere in the complex energy plane. However, functions we analyze in reality may and do contain other poles for $\omega \neq \omega_0$, so if we iterate this procedure we can without loss of generality write down:

$$T(\omega) = \sum_{i=1}^k \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega); \quad a_{-1}^{(i)}, \omega_i, \omega \in \mathbb{C}. \quad (2)$$

where k is number of poles. $a_{-1}^{(i)}$ and ω_i are residua and pole positions for i -th pole respectively, and $B^L(\omega)$ is a function regular in all $\omega \neq \omega_i$.

This approach has been already investigated, but the freedom in choosing the exact analytic form of the background contribution $B^L(\omega)$ has been introducing severe ambiguities.

*Electronic address: alfred.svarc@irb.hr

TABLE I: Toy model parameters and fitted parameters. Input parameters are given in boldface, and results of a fit in normal font. Table is given in GeV units, and $\Gamma_i = -2W_i$.

C_1	C_2	B_1	B_2	r_1	g_1	M_1	Γ_1	r_2	g_2	M_2	Γ_2	α	x_P	N_1	β	x_Q	N_2	γ	x_R	N_3	$10^2\chi_R^2$
Toy-model																					
				0.1	0.09	1.65	0.165	0.09	0.06	2.25	0.2										
Fitted results																					
Strategy a.																					
1	0	0	0	0.100	0.089	1.649	0.165	0.090	0.060	2.249	0.200	2.48	0.97	5							0.03
0	1	0	0	0.099	0.090	1.650	0.165	0.090	0.060	2.249	0.199	3.97	3.97	5							0.01
0	0	1	1	0.098	0.091	1.650	0.165	0.090	0.060	2.250	0.200	1.19	-14.94	7							0.2
0	0	-1	-1	0.099	0.089	1.649	0.1649	0.089	0.059	2.249	0.199	0.99	-9.63	7							0.01
1	0	1	1	0.103	0.100	1.653	0.171	0.101	0.067	2.249	0.221	0.71	-0.23	11							28
1	0	1	1	0.099	0.090	1.650	0.164	0.089	0.060	2.250	0.199	-2.04	-17.58	5	4.27	0.97	5				0.28
1	0	-1	-1	0.097	0.087	1.651	0.161	0.090	0.060	2.250	0.201	0.90	-0.39	20							22.0
1	0	-1	-1	0.099	0.089	1.649	0.164	0.090	0.059	2.249	0.199	2.96	-8.97	6	1.56	0.97	6				1.00
0	1	1	1	0.107	0.088	1.646	0.166	0.093	0.048	2.239	0.197	2.06	-0.89	10							114.79
0	1	1	1	0.099	0.090	1.650	0.165	0.090	0.060	2.250	0.200	1.94	-16.33	5	6.42	3.97	5				0.02
0	1	-1	-1	0.090	0.086	1.651	0.156	0.095	0.058	2.248	0.202	0.969	-0.37	12							238.38
0	1	-1	-1	0.099	0.090	1.650	0.165	0.090	0.060	2.250	0.200	0.81	-7.89	8	1.24	3.97	8				0.06
1	1	1	1	0.085	0.102	1.663	0.171	0.087	0.075	2.262	0.216	1.09	-2.64	10							328.19
1	1	1	1	0.098	0.086	1.650	0.161	0.095	0.058	2.247	0.199	0.44	-0.47	9	1.95	3.97	8				70.37
1	1	1	1	0.099	0.090	1.650	0.164	0.089	0.061	2.251	0.200	4.19	-22.99	5	2.22	3.98	5	1.67	0.97	3	0.24
1	1	-1	-1	0.090	0.105	1.657	0.182	0.078	0.061	2.260	0.189	1.38	-3.12	10							467.54
1	1	-1	-1	0.095	0.098	1.654	0.173	0.086	0.061	2.254	0.198	0.61	-0.20	9	25.91	3.98	8				60.94
1	1	-1	-1	0.100	0.090	1.650	0.165	0.090	0.060	2.250	0.200	1.85	-6.25	3	16.36	3.97	3	1.32	0.98	3	0.72
Strategy b.																					
1	0	0	0	0.069	-0.111	1.647	0.155	0.081	0.055	2.252	0.193	0.96	0.994	18							0.83
0	1	0	0	-0.101	-0.084	1.649	0.165	-0.088	-0.061	2.249	0.200	0.90	3.94	9							0.33
0	0	1	1	0.329	0.247	1.649	0.165	-0.077	0.076	2.249	0.200	1.00	-0.76	8							0.01
0	0	-1	-1	0.160	0.158	1.649	0.165	0.235	0.109	2.249	0.200	0.65	-8.41	8							0.01
1	0	1	1	0.114	-0.049	1.657	0.156	-0.146	0.418	2.254	0.187	0.52	0.00	11							12.3
1	0	1	1	-0.452	-0.036	1.649	0.165	0.096	-0.064	2.249	0.200	1.00	-5.81	5	2.06	0.97	5				0.01
1	0	-1	-1	-0.226	0.160	1.645	0.166	0.017	0.110	2.249	0.208	0.34	-0.02	20							7.12
1	0	-1	-1	0.116	-0.010	1.650	0.164	0.226	-0.202	2.249	0.199	2.14	-0.18	6	1.70	0.98	6				0.03
0	1	1	1	0.320	0.060	1.643	0.166	0.036	0.111	2.244	0.229	1.00	-0.31	10							43.59
0	1	1	1	-0.096	-0.092	1.650	0.165	-0.090	-0.059	2.250	0.200	1.11	-3.81	5	1.60	3.97	5				0.01
0	1	-1	-1	0.062	0.143	1.653	0.184	0.202	0.329	2.268	0.225	0.85	-0.05	9							102.28
0	1	-1	-1	0.101	0.291	1.650	0.165	0.090	0.062	2.250	0.199	4.25	-66.59	6	20.03	3.97	6				0.01
1	1	1	1	0.052	-0.092	1.656	0.143	0.239	0.282	2.235	0.179	1.18	-0.25	12							46.02
1	1	1	1	-0.058	-0.122	1.662	0.167	-0.054	-0.081	2.258	0.185	0.71	-0.72	6	1.38	4.00	6				25.22
1	1	1	1	-0.318	0.258	1.648	0.165	0.073	-0.104	2.247	0.207	0.67	-9.20	7	0.17	3.99	7	0.18	1.00	7	1.09
1	1	-1	-1	0.098	0.365	1.640	0.164	0.126	0.061	2.247	0.118	1.66	-1.16	12							25.63
1	1	-1	-1	-0.012	0.375	1.649	0.165	0.080	0.077	2.251	0.200	1.36	0.92	8	2.62	3.98	7				0.53
1	1	-1	-1	-0.088	0.069	1.65	0.164	-0.015	0.370	2.249	0.201	1.11	-1.62	7	3.53	4.02	7	1.56	0.97	5	0.23

The novelty of our approach is that we propose to avoid discussing the arbitrariness in all possible choices for the background function $B^L(\omega)$ by replacing it with Pietarinen expansions in power series using a complete set of functions with well known analytic properties.

Pietarinen series

If $F(\omega)$ is a general, unknown analytic function having a cut starting at $\omega = x_P$, then it can be represented in a power series of Pietarinen functions in the following way:

$$\begin{aligned} F(\omega) &= \sum_{n=0}^N c_n Z(\omega)^n, \quad \omega \in \mathbb{C} \\ Z(\omega) &= \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}, \quad c_n, x_P, \alpha \in \mathbb{R}, \end{aligned} \quad (3)$$

with the α and c_n being tuning parameter and coefficients of Pietarinen function $Z(\omega)$ respectively.

The Pietarinen series have been proposed and introduced by Ciulli [11] and Pietarinen [12], and have been with great success used in Karlsruhe-Helsinki partial wave analysis [13] when invariant amplitudes have been expanded in as many as 50 terms. The essence of the approach is the fact that a set $(Z(\omega)^n, n = 1, \infty)$ forms a complete set of functions defined on the unit circle in the complex energy plane having branch cut starting at $\omega = x_P$, but the analytic form of the function is at the beginning yet undefined. The final form of the analytic function $F(\omega)$ is obtained by introducing the rapidly convergent power series with real coefficients, and the degree of the expansion is automatically determined in fitting the input data.

The application of Pietarinen series to scattering theory

The analytic structure of each partial wave is a well known fact. Each partial wave contains poles which parameterize resonant contributions, it has cuts in the physical region starting at thresholds of elastic and all possible inelastic channels, and finally there are t-channel, u-channel and nucleon exchange contributions quantified with corresponding negative energy cuts. However, explicit analytic form of each cut contribution is not known. Instead of guessing the exact analytic form of all of them, we propose to use one Pietarinen series to represent each cut, and the number of terms in Pietarinen series will be determined by the quality of fit to the input data. So, in principle we have one Pietarinen series per cut, the branch-points x_P, x_Q, \dots are known from physics, and coefficients are determined by fitting the input data coming from real physical process. In practice, we have too many cuts (especially in the negative energy range), so we reduce their number by dividing them in two categories: all negative energy cuts are approximated with only one, effective negative energy cut represented with one Pietarinen (we usually denote its branchpoint as x_P), and each physical cut

is represented with its own Pietarinen series with branch-points determined by the physics of the process (x_Q, x_R, \dots).

So, the set of equations which define Laurent expansion+Pietarinen series method (L+P method) is:

$$\begin{aligned} T(\omega) &= \sum_{i=1}^k \frac{a_{-1}^{(i)}}{\omega_i - \omega} + B^L(\omega) \\ B^L(\omega) &= \sum_{n=0}^M c_n Z(\omega)^n + \sum_{n=0}^N d_n W(\omega)^n + \dots \\ Z(\omega) &= \frac{\alpha - \sqrt{x_P - \omega}}{\alpha + \sqrt{x_P - \omega}}; \quad W(\omega) = \frac{\beta - \sqrt{x_Q - \omega}}{\beta + \sqrt{x_Q - \omega}} + \dots \\ &a_{-1}^{(i)}, \omega_i, \omega \in \mathbb{C} \\ &c_n, x_P, d_n, x_Q, \alpha, \beta, \dots \in \mathbb{R} \\ &\text{and } k, M, N, \dots \in \mathbb{N}. \end{aligned} \quad (4)$$

As our input data are on the real axes, the fit is performed only on this dense subset of the complex energy plane. All Pietarinen parameters in set of equations (4) are determined by the fit.

Let us observe that the class of input functions which are convenient to be analyzed with this method is quite wide. One may either fit partial wave amplitudes obtained from any of theoretical models, or even experimental data directly. In either case, T-matrix is represented by the set of equations (4), minimization function is defined (usually χ^2 type), and fitting is done.

Let us summarize our fitting procedure

First of all let us observe that in the strict spirit of the method, physical Pietarinen branch points x_Q, x_R, \dots **should not** be fitting parameters. Namely, as we have declared that each known cut should be represented by its own Pietarinen series having the analytic structure of the very cut, Pietarinen branch points should be fixed to known physical branch points. And this indeed is so in the ideal case. However, in realistic case the situation is very similar to the background situation. Namely, we can never include all physical cuts in the multi-channel process, so instead of taking them all, we represent them by a smaller subset. So, for physical energy range too, Pietarinen branch points x_Q, x_R, \dots are not constants; we have to relax them and allow them to vary as fitting parameters. Later on in this paper we shall demonstrate that when we do it, physical branch points still naturally converge towards branch-points which belong to channels which dominate certain partial wave, but do not actually correspond to them exactly. The proximity of the fit results to exact physical branch points describes the "goodness of the fit, namely it tells us how well certain combination of thresholds indeed is approximating certain partial wave. And this, together with the choice of the degree of Pietarinen polynomial represents the model dependence of our method. We have, of course, never claimed that our

method is model independent because there is no such thing as model independence. However, the method fixes its model dependence naturally, by fitting to the real data. It chooses the simplest function with the given analytic properties which fit the data, and increases the complexity of the function only when the actual data require so. We shall later on demonstrate that for $1/2^-$ partial wave (S_{11}) the fit chooses πN elastic and η production branch point, while for $1/2^+$ partial wave (P_{11}) instead of choosing η production branch point, it settles close to the $\pi\pi N$ threshold which is dominant for this partial wave. Let us observe that we are still limited to the stable two-body channel representation (real branch point), so quasi stable two body channels like ρN and σN are still not included. L+P method can do it by allowing the branch points x_Q, x_R, \dots to become complex numbers, but it simply has not been done yet.

We first start with minimal number of poles, the minimal number of Pietarinen functions (we choose only dominant inelastic channels), minimal number of fitting parameters $\alpha, \beta, c_n, d_n, \dots$, and Pietarinen branch point x_P, x_Q, \dots being close to actual physical branch points. We usually start with $N = 3$. The reduced χ^2 is analyzed, and the quality of the fit is visually inspected by comparing fitting function with fitted data. If the fit is unsatisfactory (reduced χ^2 is high, or fit visually does not reproduce fitted data), the number of Pietarinen parameters c_n, d_n, \dots is increased by one. The fit is repeated, and the quality of the fit is re-estimated. This procedure is continued until we have reached the sufficient number of Pietarinen terms so that we are able to reproduce the fitted data. If the quality of the fit is still unsatisfactory, we first increase the number of poles and repeat the procedure. If no improvement is achieved by increasing the number of poles, then we increase the number of Pietarinen functions by one taking the next branch-point (next physical threshold), and repeat the procedure until the agreement has been reached.

Testing the method

We have tested the method in three ways:

I) We have constructed a toy-model function imitating the physical reality as close as possible (known pole parameters, two positive energy cuts and various background contributions), and verified how well our method reproduces the input parameters;

and to see how our method works in realistic cases

II) We have used our L+P method to extract pole parameters from single-channel amplitudes of a known, published model (N(1535) $1/2^-$ amplitude of Zagreb CMB model [14]) and compared the L+P result with pole parameters from the original publication, and

III) We have used our L+P method for something

what could never be done before, we have extract pole positions from partial wave data of ref. [15] without assuming any final functional form for the scattering matrix regular part.

I) Fitting the toy model

In principle, we should have defined a toy model input data $T^{ty}(\omega_j)$ by defining a toy model function and by normally distributing its values in order to simulate the statistical nature of real measured data. However, as the main goal of this paper is to establish the validity of the approach, we have restricted our analysis to infinitely precise data by using non-distributed toy-function values, and using the statistical weight w_j of 5 %. This enables us to test the details of our theoretical assumptions, but gives unrealistically low χ^2 values. Testing the capacity and limitations of the L+P method with realistic data for the toy model is deferred to another publication.

A toy model function is constructed by assuming a typical analytic structure of a partial wave: it is constructed as a sum of two poles, two physical cuts and several non-resonant background contributions. The function representing physical cuts is constructed from a function $\Re(x, a) = \sqrt{x^2 - 4ax}/2x$ having a cut starting from $x_0 = 4a$ on the real axes¹, and the analyticity is imposed through the once subtracted dispersion relation $\Phi(x, a) = \frac{x-x_0}{\pi} \int_{x_0}^{\infty} \frac{\Re(x', a)}{(x'-x)(x'-x_0)} dx'$. However, to simplify the demonstration of usability of L+P method, we replace all negative energy cuts with two poles deep in unphysical region. In spite of looking rather restrictive, such an approximation is fairly justified. Namely, we know that each cut can be represented by the infinite sum of poles, and as negative cut is indeed very far from the region of interest, replacing it with only two out of infinite number of poles is a good approximation (see Cutkosky CMB approach [4]).

So our toy -model function is given as:

$$\begin{aligned} T^{ty}(\omega) &= \sum_{i=1}^2 \frac{r_i^{ty} + i g_i^{ty}}{M_i^{ty} - \omega - i W_i^{ty}} + \\ &+ C_1 \Phi(\omega, 0.25) + C_2 \Phi(\omega, 1.) + B^{ty}(\omega), \\ \Phi(\omega, a) &= \frac{\sqrt{\omega(-4a + \omega)}}{2\pi\omega} \ln \frac{2a - \omega - \sqrt{\omega(-4a + \omega)}}{2a} \\ B^{ty}(\omega) &= B_1 \frac{10.}{-10. - \omega - i 5.} + B_2 \frac{10.}{-6. - \omega - i 4.}, \end{aligned} \quad (5)$$

where

$$r_i^{ty}, g_i^{ty}, M_i^{ty}, W_i^{ty} \in \mathbb{R}.$$

¹ The type of the function used for physical cuts comes from the phase space factor for two body reactions $\phi(s) = \sqrt{\Lambda(s)}/2s$, with $\Lambda(s) = s^2 - 2s(M^2 + m_\pi^2) + (M^2 - m_\pi^2)^2$, and taking $m_\pi = M$.

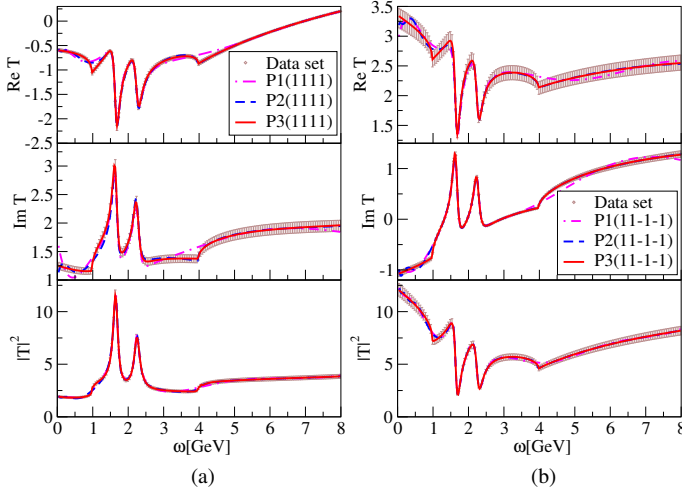


FIG. 1: **Toy model function.** In figures a) and b) we give toy-model data for the function with two poles and two cuts for two choices of background parameters, for $(B_1 = 1, B_2 = 1)$ and $(B_1 = -1, B_2 = -1)$ respectively. Dashed-dotted, dashed and full lines P1, P2 and P3 give the quality of the fit for solutions with one, two and three Pietarinen expansions respectively.

Toy model parameters for all test cases are chosen to resemble physical reality as much as possible, and are given in Table I with bold face characters.

We have applied two fitting strategies:

a) fitting both, real and imaginary part of the toy-model data (imitating the physical situation when a complete experiment is performed, and the full T-matrix is known); and

b) fitting only absolute value of the toy-model function (imitating physical situation when only incomplete data set is available, and these are usually differential cross section data).

Minimization function for case a):

$$\chi^2 = \chi_f + \lambda \chi_{Pen};$$

$$\chi_f = \sum_{j=1}^{N_{pts}} |T^{ty}(\omega_j) - T(\omega_j)|^2 / w_j^2. \quad (6)$$

Minimization function for case b):

$$\chi^2 = \chi_f + \lambda \chi_{Pen};$$

$$\chi_f = \sum_{j=1}^{N_{pts}} ||T^{ty}(\omega_j)|^2 - |T(\omega_j)|^2| / w_j^2. \quad (7)$$

In both cases w_j is corresponding statistical weight and $\chi_{Pen} = \sum_{k=1}^N c_k^2 k^3$ is Pietarinen series penalty function [12] which guarantees that the number of power-series

terms is minimal. Coefficient λ is determined in a fit, and serves to match the size of χ_f versus χ_{Pen} contributions.

In situation a) we expect a unique solution, and in situation b) we expect a full set of solutions because the relative phases of the fitted functions are still undetermined.

Let us introduce a notation $[a,b,c,d]$ meaning by definition: $(C_1 = a, C_2 = b, B_1 = c, B_2 = d)$.

We have tested the validity of the model for two backgrounds labeled $A=[a,b,1,1]$ and $B=[a,b,-1,-1]$ which strongly contribute to the toy-function strength in the observed resonance range ($1 \leq \omega \leq 3$ GeV), and produce drastically different form of the toy-function absolute value [see Fig. (1)]. We have decided to characterize the type of the background according to the absolute value shape (what basically corresponds to the differential cross section): while background A produces typical "two peak" resonance structure, background B produces a very atypical behavior in the first-second resonance region. If these numbers were differential cross sections extracted from experiment, one could not easily say whether they in case of background B represent a resonance, or some other interference effect. Therefore, we believe that these two backgrounds are the worst case scenario for the L+P method to separate resonance and background contributions.

Results for fitting strategy a)

With toy model data we tested the functionality of the model for various combinations of toy-model ingredients. The aim is to verify basic concepts of L+P method for the ideal data set. All results of the toy-model L+P fit are shown in Table I. Toy model function with all ingredients included, and for two different background contributions $(B_1 = 1, B_2 = 1)$ and $(B_1 = -1, B_2 = -1)$ is depicted in Figs. (1 a. and b.).

Two poles, no background one cut

We fit the toy model data generated by the toy function with $[1,0,0,0]$ and $[0,1,0,0]$. We establish that one Pietarinen expansion in L+P formalism is sufficient to reproduce the input data (see low χ^2 value). However, Pietarinen branch point x_P differs for both solutions (1 and 4) indicating that the cut structure is reproduced. Both, residua and pole positions are perfectly reproduced.

Two poles, background, no cuts

We fit the toy model data generated by the toy function with $[0,0,1,1]$ and $[0,0,-1,-1]$. We establish that one Pietarinen expansion in L+P formalism with cut at $x_P \ll 0$ is sufficient to reproduce the input data (see low χ^2 value). Pietarinen branch point x_P differs for both solutions indicating that the different backgrounds are fitted. Both, residua and pole positions are perfectly reproduced.

Two poles first cut, first background

We fit the toy model data generated by the toy function

with $[1,0,1,1]$ (background A). We see that for one Pietarinen expansion reduced χ^2 value is high, and the Pietarinen branch point is negative. To improve the fit we had to introduce second Pietarinen series. The fit is improved, and the fitting has converged. So, in spite of the fact that we have represented the negative cut as a sum of unphysical poles, we still see it as a cut, so we need two Pietarinen expansions to take both cuts into account. While in case of one Pietarinen the branchpoint is negative, in case of two Pietarinen one branch point is negative, and second one converges towards 1 what is the toy-function branch point. Both, residua and pole positions are perfectly reproduced.

Two poles first cut, second background

We fit the toy model data generated by the toy function with $[1,0,-1,-1]$ (background B). The situation of as for first background $[1,0,1,1]$ is reproduced. We again need two Pietarinen series. Overall conclusion is that regardless of the type of background, L+P formalism works. Both, residua and pole positions are perfectly reproduced.

Two poles second cut, both backgrounds

We fit the toy model data generated by the toy function with $[0,1,1,1]$ and $[0,1,-1,-1]$. Situation is exactly reproduced as for first cut, but with one difference: Pietarinen branchpoint x_P for two Pietarinen expansions converges towards $x_P = 4$ what is exactly second toy-model cut. Both, residua and pole positions are perfectly reproduced.

Two poles two cuts, first background

We fit the toy model data generated by the toy function $[1,1,1,1]$ first with only one Pietarinen expansion, and afterwards with two. In both cases reduced χ^2 value is, as expected, high. The value of χ^2 is better for two Pietarinen, but the result is still unsatisfactory. Adding third Pietarinen fixes the problem. The fit has converged, and Pietarinen cuts are consistent with expectations: first one is negative, and second and third are close to 1 and 4, namely close to toy-model branch-points. Both, residua and pole positions are perfectly reproduced.

Two poles two cuts, second background

Conclusions are identical as for the first background. Consequently, the L+P formalism is invariant with respect to the relative size and phase between background and physical contributions.

Results for fitting strategy b)

All conclusions as for fitting strategy a) are reproduced with one major exception:

we only reproduce pole positions, residua are quite arbitrary. As a matter of fact, when fitting strategy b) was used, we have been able to produce a whole class of solutions with almost identical χ^2 and very different residua, exactly what was to be expected.

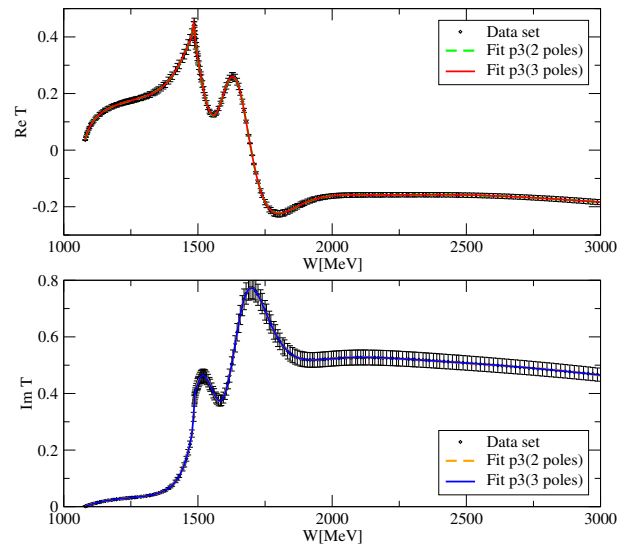


FIG. 2: **Zagreb S11**. Dashed and full lines gives the quality of the fit for two and three pole solutions in both cases with three Pietarinen series.

II) Fitting πN elastic $N(1535)$ $1/2^-$

To illustrate how usable L+P method is in reality, we have decided to fit one realistic set of input data. We have taken πN elastic $N(1535)$ $1/2^-$ partial wave from Zagreb CMB model, and fitted it with L+P method. We have chosen fitting strategy a) and fitted both, real and imaginary part. We have to mention that we did not have much use of the Penalty function in fitting the toy-model, but for realistic data from Zagreb CMB using the penalty function was unavoidable. Results are shown in Table II and Fig. (3).

We know that πN elastic scattering has at least two branch-points in the physical region (πN elastic threshold in Zagreb model at $w = 1.076$ GeV and η production in Zagreb model at $w = 1.486$ GeV), background contribution, and at least two poles. So we know that our L+P solution should have minimally two poles and three Pietarinen series. Therefore, we have started the fitting procedure with two poles, and three Pietarinen series, and obtained a reasonable solution. The solution was acceptable visually, and by χ^2 . Two Pietarinen branch-points came out very close to physical thresholds, the third Pietarinen branch-point came out far in the negative energy plane describing the background contribution from the negative cut, and pole parameters have been quite close to parameters obtained in analytic continuation. We wondered if the fit could be improved by increasing the number of poles by one, so we have tried a three pole-three Pietarinen fit. The reduced χ^2 was somewhat improved, the pole parameters came somewhat closer to the analytic continuation value, but in general the third resonance was uncertain and rather wide. It is interesting to observe that both physical Pietarinen

TABLE II: Parameters of the fit for the πN elastic $N(1535)$ $1/2^-$ partial wave from Zagreb CMB model. Table is given in MeV units, and $\Gamma_i = -2W_i$.

	M_1	Γ_1	M_2	Γ_2	M_3	Γ_3	x_P, N_1	x_Q, N_2	x_R, N_3	$10^2 \chi^2_R$
Zagreb CMB	1521(14)	190(28)	1646(8)	204(17)	1790(26)	420(45)				
Fit: $\begin{smallmatrix} 2 \text{ poles} \\ 3 \text{ Pietarinen} \end{smallmatrix}$	1525	120	1653	189	-	-	-696, 15	1058, 15	1484, 15	2.53
Fit: $\begin{smallmatrix} 3 \text{ poles} \\ 3 \text{ Pietarinen} \end{smallmatrix}$	1529	146	1647	192	1801	2321	933, 15	1057, 15	1482, 15	1.31

branch-points remain where they were (close to physical thresholds), but the third came out much higher. So, L+P method conclusively gives two resonances in Zagreb CMB amplitudes, but indicates the existence of the third one which is not well defined.

We ended up with using three Pietarinen series with one threshold in unphysical range representing the Pietarinen series for background contributions, and two thresholds in the physical range. One was at $x_Q = 1.057$ GeV (near physical threshold), and second at $x_R = 1.482$ GeV (near η production threshold). Pole positions of Zagreb model are almost perfectly reproduced. Using L+P method we obtained three poles: $M_1 = 1.529 - i0.073$, $M_2 = 1.647 - i0.096$ and $M_3 = 1.801 - i1.160$ GeV, what is to be compared to Zagreb CMB poles from ref. [14]: $M_1^{Zg} = 1.521 - i0.095$, $M_2^{Zg} = 1.646 - i0.102$ and $M_3^{Zg} = 1.790 - i0.210$ GeV. First two poles agree almost ideally, the existence of third pole around 1.8 GeV is allowed, but single-channel data are insufficient to pin it down more precisely as expected from coupled-channel calculation.

As Zagreb CMB analytic continuation was performed in three channel model, we conclude that two resonances is the best what a single-channel method like L+P can give. We also allow for some deviations in quantitative values of two well established resonances, since L+P is, again, only single-channel model. A correct recipe would be to repeat the L+P procedure on amplitudes from all channels, and then make an analysis. Extending the analysis to other channels might shift pole position somewhat deeper in the complex energy plane, single channel analysis as we did it now is, however, "satisfied" with poles being somewhat closer to the real axes.

III) Poles from experiment: Fitting GWU-SES partial wave data

The novelty of L+P method is that it allows extracting poles from data coming directly from experiment, i.e. to analyze the numbers which are obtained using only statistical methods with very little underlying theory. We shall illustrate this feature on GWU single energy (SES), and energy dependent (SP06) solutions [15].

We have extracted pole positions from both, GWU-SES ("experiment") and energy dependent SP06 solution which is obtained from GWU-SES with the theoretical analysis based on using the polynomial form of Chew-Mandelstam K-matrix model, and compared results.

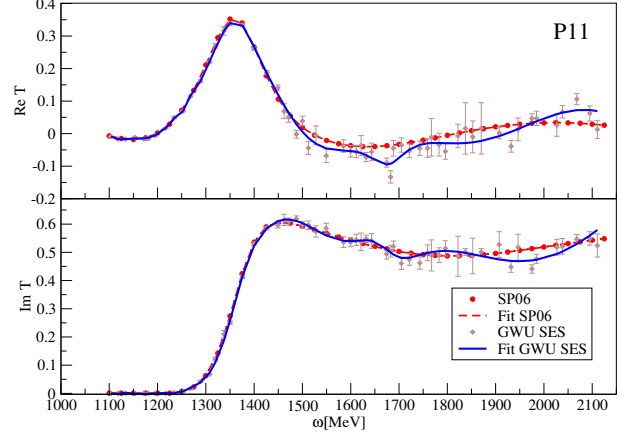


FIG. 3: GWU SP06 and SES.

Let us point out that GWU group is able to extract poles from their SES only through creating their energy dependent SP06 solution, and analytically continuing it into complex energy plane. Our method allows us to fit both, SES and SP06, and compare results.

We came to a very interesting conclusion:

We can fit both, GWU-SES and SP06, by using two poles and three Pietarinen series, first threshold being in unphysical range, second near physical threshold and third near $\pi\pi N$ threshold (observe that S_{11} was fitted with Pietarinen near η production threshold). Results of a fit are given in Table III. In boldface we give pole positions from original publication. The original publication finds Roper poles on two Riemann sheets (on P[221] and on P[121]), and we give them both. However, our L+P method is restricted to the first Riemann sheet only.

From Table III we see that for SP06 the second pole in our fit is completely undetermined, while for GWU-SES it is definitely established, and close to 1.7 GeV. And that is very similar to what GWU group claims: they claim that they do not need $P_{11}(1710)$ in their analysis at all. However, their analysis is based solely on energy dependent analysis. We agree with them in a sense that we also do not need $P_{11}(1710)$ to fit their energy dependent solution SP06. However, when fitting GWU-SES what they could not do, our fit **REQUIRES** the presence of the $P_{11}(1710)$ state! Our results are: $M_1^{\text{SP06}} = 1.358 - i0.094$, $M_2^{\text{SP06}} = 2.694 - i0.893$ GeV for SP06 (second resonance completely undetermined), and

TABLE III: GWU SES and SP06 from ref. [15]. Poles from original publication are given in boldface. Table is given in GeV units, and $\Gamma_i = -2W_i$.

	r_1	g_1	M_1	Γ_1	r_2	g_2	M_2	Γ_2	α	x_P	N_1	β	x_Q	N_2	γ	x_R	N_3	χ_R^2
GWU SES	-	-	-	-	-	-	-	-										
GWU SP06	-	-	1.388 1.358	164 162	-	-	-	-										
L+P SES	0.008	-0.059	1.358	0.156	-0.001	0.006	1.679	0.106	1.109	-2.409	15	0.573	1.074	15	0.677	1.243	15	1.815
L+P SP06	0.001	-0.065	1.357	0.188	0.180	0.504	2.694	1.786	0.987	-1.195	15	1.125	1.024	15	0.711	1.208	15	0.005

$M_1^{\text{SES}} = 1.358 - i 0.078$, $M_2^{\text{SES}} = 1.679 - i 0.053$ GeV for SES (second resonance firmly established).

So, our L+P method allows us to explicitly see that their SES contain $P_{11}(1710)$, and their energy dependent analysis smeared it out. Results of a fit are given in Fig. 3.

Let us also observe that taking only two Pietarinen series for physical range is still an approximation. Namely, in both fits given in Table III second and third threshold are close, but not exactly on top of first two physical thresholds. The first threshold for SES is very close to πN channel opening (at 1.076 GeV), while the second one is notably above $\pi\pi N$ threshold (at 1.218 GeV). For SP06 fit thresholds show opposite trend - first threshold is significantly lower, and second threshold is very close to $\pi\pi N$ value. That indicates the level of precision of our pole results, and stimulates further improvement of the model. However, even this level of precision is sufficient for pursuing the main argument .

Conclusion

We propose a new single-channel pole extraction method based on Laurent decomposition and Pietarinen series. Instead of guessing the unknown T-matrix functional form as it has been done up to now, we propose to use the Laurent's expansion of single-channel partial wave T-matrices to obtain their pole positions. We make use of a fairly well known analytic structure of each partial wave, and expand the unknown regular part of the Laurent expansion in power series of Pietarinen functions using one Pietarinen series for each known cut. Finally, we fit the input data and get the simplest function for the regular Laurent part which has the correct analytic structure and together with the known number of poles reproduces the input. The resulting pole parameters are the genuine poles of our unknown input function. Even to the surprise of the authors, the fitting procedure in analyzed test cases turns out to be extremely stable and reliable regardless of the number of Pietarinen series, and extracted pole parameters turn out to be fairly confident. We warn the reader that this is a *single-channel method*, and it will recognize only those resonances which strongly couple to the observed channel. Other resonances will be only hinted at, and have to be pinned down by analyzing other channels.

Appendix

Pietarinen expansion coefficients of solutions presented in Table I (**Strategy a**).

Strategy a:				One Pietarinen expansion										
C_1	C_2	B_1	B_2	c_0 c_{11}	c_1 c_{12}	c_2 c_{13}	c_3 c_{14}	c_4 c_{15}	c_5 c_{16}	c_6 c_{17}	c_7 c_{18}	c_8 c_{19}	c_9 c_{20}	c_{10} c_{21}
1	0	0	0	1.079 —	−1.080 —	0.155 —	−0.175 —	0.044 —	−0.022 —	— —	— —	— —	— —	—
0	1	0	0	0.947 —	−1.031 —	0.205 —	−0.153 —	0.044 —	−0.012 —	— —	— —	— —	— —	—
0	0	1	1	26.944 —	62.508 —	20.310 —	−51.562 —	−20.914 —	55.819 —	58.087 —	17.230 —	— —	— —	—
0	0	−1	−1	−54.437 —	−198.58 —	−266.51 —	−95.229 —	150.39 —	212.01 —	111.12 —	22.976 —	— —	— —	—
1	0	1	1	−2.199 —	−2.705 —	−2.005 —	−1.117 —	0.402 —	1.497 —	2.562 —	2.329 —	1.827 —	0.918 —	0.424 —
1	0	−1	−1	4.011 0.900	2.215 1.289	3.875 −0.409	2.001 0.691	2.916 0.750	−0.045 1.967	0.723 1.710	−1.118 1.584	1.620 0.746	0.861 0.307	2.834 —
0	1	1	1	−2.798 —	0.641 —	−6.514 —	5.436 —	−11.528 —	8.090 —	−11.12 —	5.779 —	−5.968 —	1.872 —	−1.377 —
0	1	−1	−1	3.538 0.448	1.523 0.479	2.114 —	0.383 —	1.654 —	−0.325 —	1.1333 —	0.130 —	1.499 —	0.773 —	1.345 —
1	1	1	1	62.71 —	417.41 —	1437.3 —	3228.7 —	5218.1 —	6285.7 —	5717.3 —	3879.2 —	1894.7 —	607.32 —	101.69 —
1	1	−1	−1	88.678 —	427.56 —	1298 —	2637.8 —	4045.7 —	4716.2 —	4307.9 —	2996.2 —	1573.5 —	555.82 —	118.2 —
Strategy a:				Two Pietarinen expansions										
C_1	C_2	B_1	B_2	c_0 d_0	c_1 d_1	c_2 d_2	c_3 d_3	c_4 d_4	c_5 d_5	c_6 d_6	c_7 d_7	c_8 d_8	c_9 d_9	c_{10} d_{10}
1	0	1	1	3.849 3.849	5.812 −4.385	−3.141 4.217	−10.608 −3.331	−6.01 1.494	−0.704 −0.416	— —	— —	— —	— —	—
1	0	−1	−1	1.731 1.731	−2.290 −1.212	−0.002 0.573	−5.747 −0.402	0.693 0.194	−2.229 −0.070	1.239 0.018	— —	— —	— —	—
0	1	1	1	18.667 18.667	117.70 −8.027	189.14 8.294	171.86 −5.408	89.342 2.066	22.696 −0.441	— —	— —	— —	— —	—
0	1	−1	−1	−32.588 −32.588	−217.23 −0.676	−220.56 0.347	30.972 −0.195	174.4 0.109	−5.098 −0.055	−190.34 0.025	−147.91 −0.008	−37.90 0.002	— —	—
1	1	1	1	−1.255 −1.255	−2.575 −0.431	1.651 −0.754	4.307 1.189	3.093 −1.122	−1.463 0.772	−2.319 −0.403	0.232 0.144	1.907 −0.032	1.221 —	— —
1	1	−1	−1	−14.963 −14.963	3.089 136.57	4.009 −230.17	3.351 313.67	3.146 −443.08	1.1584 519.08	0.1584 −476.42	−0.686 301.89	−0.572 −87.63	−0.430 —	— —
Strategy a:				Three Pietarinen expansions										
C_1	C_2	B_1	B_2	c_0 d_0 e_0	c_1 d_1 e_1	c_2 d_2 e_2	c_3 d_3 e_3	c_4 d_4 e_4	c_5 d_5 e_5	c_6 d_6 e_6	c_7 d_7 e_7	c_8 d_8 e_8	c_9 d_9 e_9	c_{10} d_{10} e_{10}
1	1	1	1	2.275 2.274 2.277	19.465 −0.616 −0.878	27.86 0.044 0.063	35.107 −0.023 −0.029	18.749 0.019 —	12.32 −0.007 —	— — —	— — —	— — —	— — —	—
1	1	−1	−1	10.107 10.107 10.107	1.372 −70.87 −0.757	0.849 65.949 0.077	−0.515 −21.659 −0.017	— — —	— — —	— — —	— — —	— — —	— — —	—

Pietarinen expansion coefficients corresponding to solutions presented in Table I (**Strategy b**).

Strategy b:				One Pietarinen expansion										
C_1	C_2	B_1	B_2	c_0 c_{11}	c_1 c_{12}	c_2 c_{13}	c_3 c_{14}	c_4 c_{15}	c_5 c_{16}	c_6 c_{17}	c_7 c_{18}	c_8 c_{19}	c_9 c_{20}	c_{10} c_{21}
1	0	0	0	0.676 -2.938	-0.795 2.575	0.947 -1.934	-1.948 1.520	2.450 -0.864	-3.236 0.553	4.100 -0.248	-4.084 0.121	4.034 -	-3.702 -	3.640 -
0	1	0	0	-0.362 -	0.536 -	-0.410 -	0.211 -	-0.170 -	0.132 -	-0.042 -	0.074 -	-0.011 -	0.021 -	- -
0	0	1	1	-0.560 -	-1.205 -	-2.397 -	-1.364 -	-0.517 -	-1.170 -	-0.612 -	-0.193 -	-0.199 -	- -	- -
0	0	-1	-1	0.564 -	-0.020 -	1.994 -	1.060 -	0.168 -	2.039 -	0.397 -	0.598 -	1.357 -	- -	- -
1	0	1	1	1.255 0.245	1.799 -	2.300 -	1.412 -	1.418 -	2.447 -	1.143 -	1.178 -	1.134 -	0.484 -	0.300 -
1	0	-1	-1	0.777 1.029	0.986 0.987	1.832 0.782	1.284 1.281	1.641 1.237	0.661 0.316	0.807 1.570	1.056 1.216	0.709 -0.781	1.583 -0.934	0.617 -
0	1	1	1	1.388 -	1.753 -	5.598 -	6.395 -	11.384 -	10.090 -	13.046 -	9.156 -	9.852 -	3.338 -	2.805 -
0	1	-1	-1	1.565 -	-0.031 -	-0.235 -	1.319 -	1.635 -	1.836 -	1.316 -	1.179 -	0.392 -	0.381 -	- -
1	1	1	1	-0.088 1.242	1.193 0.037	-0.502 -	2.507 -	-0.769 -	3.256 -	-1.326 -	5.019 -	-0.963 -	2.993 -	-0.645 -
1	1	-1	-1	4.426 2.423	-1.681 -1.484	3.943 -	-3.801 -	3.305 -	0.263 -	-0.650 -	6.418 -	-4.716 -	6.584 -	-4.286 -
Strategy b:				Two Pietarinen expansions										
C_1	C_2	B_1	B_2	c_0 d_0	c_1 d_1	c_2 d_2	c_3 d_3	c_4 d_4	c_5 d_5	c_6 d_6	c_7 d_7	c_8 d_8	c_9 d_9	c_{10} d_{10}
1	0	1	1	0.512 0.512	1.761 -0.467	1.072 0.104	-0.373 -0.375	1.552 0.148	1.780 -0.039	- -	- -	- -	- -	- -
1	0	-1	-1	0.301 0.301	-2.718 0.455	3.135 -2.287	0.373 -1.714	-3.393 4.815	1.769 -3.150	2.913 0.837	- -	- -	- -	- -
0	1	1	1	0.793 0.793	0.662 0.572	0.629 -0.120	1.267 0.035	1.042 -0.011	0.081 0.003	- -	- -	- -	- -	- -
0	1	-1	-1	-1.649 -1.649	-12.558 -29.437	2.551 8.936	20.807 12.801	-4.393 12.373	-17.838 -25.197	-14.688 7.424	- -	- -	- -	- -
1	1	1	1	6.410 6.410	33.756 1.223	55.745 -0.903	59.202 0.437	44.944 -0.194	21.583 0.060	6.587 -0.171	- -	- -	- -	- -
1	1	-1	-1	1.960 1.960	-2.298 -0.817	2.642 0.279	-2.579 -0.581	2.374 0.392	-1.659 0.115	1.027 -0.197	-0.424 0.068	0.142 -	- -	- -
Strategy b:				Three Pietarinen expansions										
C_1	C_2	B_1	B_2	c_0 d_0 e_0	c_1 d_1 e_1	c_2 d_2 e_2	c_3 d_3 e_3	c_4 d_4 e_4	c_5 d_5 e_5	c_6 d_6 e_6	c_7 d_7 e_7	c_8 d_8 e_8	c_9 d_9 e_9	c_{10} d_{10} e_{10}
1	1	1	1	-1.608 -1.608 -1.608	-5.203 0.515 1.518	5.819 0.252 1.380	1.520 -1.565 -0.229	-3.035 -2.579 -0.321	4.982 -2.229 0.411	5.711 -1.062 0.598	-0.305 -0.267 0.282	- -	- -	- -
1	1	-1	-1	1.842 1.842 1.842	19.214 11.827 -5.396	40.571 24.661 17.925	49.783 -54.662 -20.529	42.630 49.917 11.084	32.165 -30.450 -2.477	6.188 12.023 -	3.091 -2.268 -	- -	- -	- -

-
- [1] International Workshop on NEW PARTIAL WAVE ANALYSIS TOOLS FOR NEXT GENERATION HADRON SPECTROSCOPY EXPERIMENTS, June 20-22, 2012, Camogli, Italy. [www.ge.infn.it/~athos12].
 - [2] M. Döring, C. Hanhart, F. Huang, S. Krewald, and U.-G. Meissner, Nucl. Phys. **A829**, 170 (2009), and references therein.
 - [3] B. Juliá-Díaz, H. Kamano, T.-S. H. Lee, A. Matsuyama, T. Sato, N. Suzuki, Phys.Rev. **C80**, 025207 (2009), and references therein.
 - [4] R. E. Cutkosky, C. P. Forsyth, R. E. Hendrick, and R. L. Kelly, Phys. Rev. **D 20**, 2839 (1979).
 - [5] M. Batinić, I. Šlaus, A. Švarc, and B. M. K. Nefkens, Phys. Rev. **C 51**, 2310 (1995); M. Batinić et al., Phys. Scr. **58**, 15 (1998).
 - [6] A. V. Anisovich, R. Beck, E. Klempt, V. A. Nikonov, A. V. Sarantsev, U. Thoma, Eur.Phys.J. **A48**, 15 (2012), and references therein.
 - [7] G. Höhler, π N Newsletter **9**, 1 (1993).
 - [8] N.G. Kelkar, M. Nowakowski, Phys.Rev. **A78**, 012709 (2008), and references therein.
 - [9] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
 - [10] S. Ceci, J. Stahov, A. Švarc, S. Watson, and B. Zauner, Phys. Rev. **D 77**, 116007 (2008).
 - [11] I. Ciulli, S. Ciulli, and J. Fisher, Nuovo Cimento **23**, 1129 (1962).
 - [12] E. Pietarinen, Nuovo Cimento Soc. Ital. Fis. **12A**, 522 (1972).
 - [13] G. Höhler, *Pion Nucleon Scattering*, Part 2, Landolt-Bornstein: Elastic and Charge Exchange Scattering of Elementary Particles, Vol. 9b (Springer-Verlag, Berlin, 1983).
 - [14] M. Batinić, S. Ceci, A. Švarc, and B. Zauner, Phys. Rev. **C 82**, 038203 (2010).
 - [15] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky, and R. L. Workman, Phys. Rev. **C 74**, 045205 (2006); [<http://gwdac.phys.gwu.edu/analysis/pin.analysis.html>]